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Cnoidal waves and bores in uniform channels of arbitrary cross-section

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A steady nonlinear dispersive wave theory is developed in terms of three important invariants of channel flow: discharge, energy, and momentum flux. As such, the work is an extension of Benjamin & Lighthill's approach for rectangular channels.

Considering the differential equation obtained, we examine the behaviour of flows and wave systems in arbitrary channels for changes of energy and momentum. In particular, the bore problem is studied, and previous approaches to this problem, using linear wave theory, are seen to be invalid. The present theory describes several phenomena of open-channel flow, explains a scatter in previously obtained experimental results, and enables simple design recommendations to be made for channels in which stationary or moving bores are expected.

While this work does describe the variation of physical quantities across the channel section, there are some important three-dimensional phenomena, noted experimentally, which remain unexplained.

1. Introduction

The study of surface gravity waves in channels of arbitrary uniform cross-section has followed a path similar to the classical studies of waves in rectangular channels, of which it is a generalization. The basic equations of motion are the same, while the boundary conditions and three-dimensional effects do not unduly complicate the problem, to first order at least.

Kelland (Kelland 1839; Lamb 1932, § 169) obtained the speed of propagation of infinitesimal long waves $c_0 = (gA_0/B_0)^{\frac{1}{2}}$, where g is the gravitational acceleration and A_0 is the cross-sectional area occupied by the undisturbed liquid, having width B_0 at the surface. The ratio A_0/B_0, the 'hydraulic mean depth', recurs frequently in all studies of wave motion in non-rectangular channels; simple as the concept may be, it appears to be the fundamental length scale which is crucial in all studies of fluid motion in these channels. In rectangular channels of depth h, Kelland's result reduces to the well-known expression obtained from simple tidal theory, $c_0 = (gh)^{\frac{1}{2}}$.

From here, studies diverge into the two standard nineteenth-century approaches for water waves. The study of infinitesimal waves, which allows the

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equations of motion to be linearized, but which makes no restriction on the velocity distribution, was initiated by Rayleigh (Rayleigh 1876; Lamb 1932, § 233, § 252) to obtain exact dispersion relationships connecting the wave speed with wavelength for waves in rectangular channels. Solutions for non-rectangular channels have only been obtained for some special cases (Lamb 1932, § 261).

The other approach is the study of disturbances which are propagated by fluid motions in which the pressure distribution is hydrostatic, giving no dispersion, while the nonlinear effects can be incorporated exactly. Lamb (1932, § 187) gives the solution for rectangular channels. For arbitrary channels, Escoffier (Escoffier & Boyd 1962) obtained the exact solution, showing that disturbances involving a local area of cross-section A and surface width B are propagated at a velocity relative to the fluid of $c = (gA/B)^{\frac{1}{2}}$. We see that, since A/B (the local hydraulic mean depth) is an increasing function of height, there is a tendency for the wave to steepen, eventually forming a bore, as obtained by Airy for rectangular channels.

For disturbances in which nonlinearity and dispersion are both small but finite (a sort of intersection of the above two theories), Korteweg & de Vries (1895) obtained an equation governing unsteady motion in rectangular channels. When the two effects balance one another, we have steady motion, for which the surface takes on a cn^2 form, giving 'cnoidal' waves, or, for the appropriate boundary condition, a solitary wave. Peregrine (1968) and Shen (1969) have obtained the equivalent unsteady equation for channels of arbitrary cross-section. Peters (1966), meanwhile, had obtained equations for the solitary wave in such channels, but where the flow was not necessarily irrotational, and Shen (1968) had obtained the unsteady equations for the case of stratified flow.

Experimental studies have been carried out by Sandover & Taylor (1962), Peregrine (1969) and by Benet & Cunge (1971). Peregrine's experiments on solitary waves in trapezoidal channels verified the results of his theory; the other workers carried out their experiments on bores in model and full-scale trapezoidal channels. These reports include (i) observations of curved wave crests when viewed along the channel; (ii) a greater tendency for the waves to break at the sides of the channel than for the rectangular case; and (iii) extra three-dimensional effects, which include curvature of the wave crest in plan, a tendency for the waves to be unsteady, and a marked 'fish-tail' pattern which obscured the main wave system for wide shallow channels. Benet & Cunge noticed a wide scatter in results.

In a study of bores in rectangular channels, Lemoine (1948) used a linear wave theory in an attempt to relate the energy loss at an undular bore to the waves created, and to use this theory to describe experimental results of Favre (1935). Benjamin & Lighthill (1954) demonstrated the invalidity of such a linear theory, showing that a nonlinear dispersive wave theory is more valid, and applied such a theory to the bore problem in a rectangular channel, showing that the waves downstream of an undular bore are cnoidal in nature. In the process they provided an explanation of the scatter of Favre's results as being due to the variability of energy loss at the bore.

Preissmann & Cunge (1967) used a linear theory similar to that of Lemoine, but generalized to include trapezoidal channels; Benet & Cunge (1971) have applied this theory to their field and experimental observations. They obtained a scatter of experimental results, similar to that of Favre.

The present work sets out to examine steady nonlinear dispersive waves in channels of arbitrary cross-section, generalizing the work of Benjamin & Lighthill, whose analysis and discussion it closely follows.

In § 2 we briefly consider existing steady wave theories. Linearized theories are examined, and it is shown that these are not applicable to the waves behind undular bores.

In § 3 we produce an expansion scheme, which, when substituted into the equations of motion, enables nonlinear dispersive solutions to be obtained. The difference between the present work and previous approaches is that the solutions are obtained in terms of the channel flow quantities: discharge, energy, and momentum flux. A differential equation is obtained, and is solved to give the equations for cnoidal waves in channels of any cross-section, in terms of the flow invariants and certain geometrical parameters of the cross-section, including the solution of a Neumann problem on the undisturbed cross-section of the channel.

Finally, in § 4 we examine the physical implications of the differential equation. Energy and momentum changes to uniform flows are seen to be able to produce wave systems, and this is applied to the case of a bore. We see that if there were no losses at the bore then we could only obtain a solitary wave; if we have the full energy loss associated with a hydraulic jump then we have uniform subcritical flow and can have no waves. When the energy loss at the bore is intermediate between these two extremes we expect a train of cnoidal waves, completely defined by the energy loss. In this way we explain the scatter of results found by Benet & Cunge: it is due to a variable energy loss at the bore, as was found by Benjamin & Lighthill for rectangular channels. The present theory is able to give some simple design recommendations for non-rectangular channels, for it shows that the highest wave possible, defining the necessary depth of channel, is the solitary wave.

This work does provide evaluations of the change of surface elevation across the channel, and as such, constitutes a three-dimensional theory. To the order of accuracy obtained, however, the theory provides no explanation of the more complicated three-dimensional effects noted in the experimental reports of Peregrine and Sandover & Taylor.

2. Equations of motion and some solutions

2.1. *Equations of motion*

Consider a steady wave system in a uniform channel of arbitrary cross-section. By superimposing a velocity on the system we have a train of stationary waves under which we have a steady flow, assumed to be incompressible, inviscid and irrotational. We set up a co-ordinate system (x, y, z) in which the waves are stationary: x is the direction of flow, y is vertically upwards and z is horizontal, perpendicular to the flow direction. The origin is at the channel invert, such that the maximum depth over the undisturbed cross-section is h.

We can define a velocity potential $\phi(x, y, z)$ because the flow is irrotational. As it is incompressible as well, we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0, \tag{2.1}$$

which holds throughout the flow.

The kinematic condition that flow does not pass through the channel boundaries is

$$\partial\phi/\partial n = 0 \tag{2.2}$$

on the solid boundaries, where n is the outward normal to the curve specified by the intersection of the channel bottom with the y, z plane.

Fluid particles on the free surface remain on the free surface, giving the kinematic condition

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial\eta}{\partial z} \tag{2.3}$$

on $y = \eta(x, z)$, the free surface. We must also have zero pressure on this surface, giving the dynamic condition

$$R = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]+g\eta, \tag{2.4}$$

where R is the energy per unit mass, which, in the absence of losses, is a constant throughout the flow; g is the gravitational acceleration.

Equations (2.1)–(2.4) constitute a system for which solutions for the dependent variables $\phi(x, y, z)$ and $\eta(x, z)$ may be sought. At this stage we define three physical quantities which must be constant throughout the flow, and in terms of which we shall subsequently obtain solutions. These are Q, the volume flow rate at any cross-section; R, the energy per unit mass at any point, equal to that defined in (2.4); and S, the momentum flux (including transfer by pressure and by convection) at any cross-section divided by the density ρ. Thus,

$$Q = \iint_A \phi_x \, dy \, dz, \tag{2.5}$$

$$R = p/\rho + gy + \tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2), \tag{2.6}$$

$$S = \iint_A (p/\rho + \phi_x^2) \, dy \, dz, \tag{2.7}$$

where A refers to the region of integration, which is the entire cross-section of fluid at each section, and $p(x, y, z)$ is the pressure at any point. Each quantity Q, R and S must be independent of the cross-section chosen if viscous effects and other losses are neglected.

2.2. *Existing solutions*

Lamb (1932, § 260) gives a linearized solution of (2.1)–(2.4). This solution is in terms of a mixed boundary-value problem on a constant domain – the cross-section of flow to which the wave system is a small disturbance. Analytical solutions to this problem have been obtained for some special cases only. Solutions for isosceles triangles, with angles of $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$ at the bottom vertex are given in Lamb (1932, § 261), while Preissmann & Cunge (1967) obtained numerical results for trapezia.

The opposite case to this, where the wave amplitude is finite, but where the wavelength is so long that dispersion effects can be ignored, has been studied by Escoffier (Escoffier & Boyd 1962). The wave speed is an increasing function of wave height, thus we have a gradual steepening of waves of elevation until a discontinuity occurs.

By analogy with waves in rectangular channels, we may show that dispersive effects are proportional to $(A_0/\lambda B_0)^2$ and that nonlinear effects are measured by aB_0/A_0, where A_0 is the undisturbed cross-sectional area of flow, B_0 is the breadth at the undisturbed water surface, and where λ and a are the wavelength and amplitude respectively. For a rectangular channel, the ratio A_0/B_0 is equal to the depth; for channels of arbitrary cross-section, A_0/B_0 is the effective depth.

Obtaining the ratio of nonlinear to dispersive effects, we see that this is measured by $a\lambda^2/(A_0/B_0)^3$. For linearized theory to apply to any situation, $a/(A_0/B_0)$ must be small, and $a\lambda^2/(A_0/B_0)^3$ must be small as well. Where waves are long, the ratio $a\lambda^2/(A_0/B_0)^3$ must be large. In cases where this ratio is of order unity, we obtain steady finite waves, where the steepening tendency is balanced by the dispersion effects. For this situation we expect to obtain waves having a form similar to cnoidal and solitary waves in rectangular channels.

In the linear theory of Preissmann & Cunge (1967), wavelengths and amplitudes of undulations behind bores in trapezoidal channels are given. On calculating values of the ratio $a\lambda^2/(A_0/B_0)^3$ from these results, we find that it varies between 4 and 23, showing the incorrectness of neglecting nonlinear effects. A nonlinear theory for undular bores is therefore required; we proceed to a theory in which both nonlinear and dispersive effects are finite but small, and are of the same order of magnitude. The order of accuracy is recognized throughout.

3. Steady nonlinear dispersive waves

3.1. *Formation of differential equation*

Now we consider the combined effects of dispersion and nonlinearity by writing an expansion for ϕ, the velocity potential, in which both are included. Thus we may write

$$\phi = f(x) - P_1(y,z)f''(x) + P_2(y,z)f^{\text{iv}}(x) - \dots, \tag{3.1}$$

in which the $f(x)$ incorporates all nonlinear effects, for which we subsequently expand in terms of amplitude, and the $P_n(y,z)$, $n = 1, 2, \dots$, give the variation over the cross-section, the nth-order dispersion term.

As this equation is to be truncated, and we wish to know the order of approximation throughout the analysis, we scale the independent variables:

$$x_1 = x/l, \quad y_1 = y/h, \quad z_1 = z/h, \tag{3.2}$$

in which l is a measure of the longitudinal extent of each wave and h is the maximum depth of liquid over the cross-section with an undisturbed free surface, the y co-ordinate of this surface with the origin at the channel invert. Because we have non-dimensionalized both y and z with respect to this, we limit our attention to channels which are not notably broad, having maximum depth the same order of magnitude as the width. In the previous sections we have seen that A_0/B_0 is a more characteristic dimension of the cross-section, however, in the equations of motion and the dependent variables we find that it is more convenient to use the physical dimension h.

We now expand the dependent variables:

$$f(x) = lc_0(x_1 + \epsilon q_1(x_1) + \epsilon^2 q_2(x_1) + O(\epsilon^3)), \tag{3.3}$$

where c_0 is the wave speed of infinitesimal long waves and $\epsilon = a/h$ is the dimensionless wave amplitude,

$$P_1(y, z) = h^2 V(y_1, z_1) \tag{3.4}$$

and

$$\eta(x, z) = h(1 + \epsilon\eta_1(x_1, z_1) + \epsilon^2\eta_2(x_1, z_1) + O(\epsilon^3)). \tag{3.5}$$

Substituting (3.3) and (3.4) into the original expansion (3.1) gives

$$\phi = lc_0(x_1 + \epsilon q_1 + \epsilon^2 q_2 - \epsilon\sigma^2 V(y_1, z_1)\, q_1'') + O(\epsilon^3, \epsilon^2\sigma^2, \epsilon\sigma^4), \tag{3.6}$$

where $\sigma = h/l$.

In § 2 we saw that, for steady nonlinear dispersive waves, $a\lambda^2/(A_0/B_0)^3 = O(1)$, thus for channels in which $hB_0/A_0 = O(1)$ we have $\epsilon/\sigma^2 = O(1)$, and the ϵ^2 and $\epsilon\sigma^2$ terms in (3.6) are to be taken as being of the same order. Similarly all the error terms in (3.6) are of the same order: we group them and subsequently show the error term only in powers of ϵ. In the retained terms, however, we must continue to distinguish between ϵ and σ^2.

Substituting these expansions for the dependent variables into the equations of motion, we group the terms according to order and obtain the following.

Laplace's equation (2.1), for incompressibility and irrotationality, gives

$$\frac{\partial^2 V}{\partial y_1^2} + \frac{\partial^2 V}{\partial z_1^2} = 1 \quad \text{at order } \epsilon, \tag{3.7}$$

to be satisfied in the undisturbed channel cross-section, non-dimensionalized in each co-ordinate direction with respect to h. Thus the upper boundary is at $y_1 = 1$ and is of width $b_0 = B_0/h$, and the domain has cross-sectional area $a_0 = A_0/h^2$. The boundary condition on the solid boundary (2.2) gives

$$\partial V/\partial n_1 = 0 \quad \text{at order } \sigma^2. \tag{3.8}$$

On the surface $y_1 = 1$, we obtain from the kinematic boundary condition (2.3)

$$\frac{\partial \eta_1}{\partial x_1} + \frac{\partial^2 q_1}{\partial x_1^2}\frac{\partial V}{\partial y_1}(1, z_1) = 0 \quad \text{at order } \sigma\epsilon, \tag{3.9}$$

while the dynamic surface boundary condition (2.4) gives, on $y_1 = 1$,

$$\eta_1 + \frac{c_0^2}{gh}\frac{\partial q_1}{\partial x_1} = 0 \quad \text{at order } \epsilon, \tag{3.10}$$

$$\eta_2 + \frac{c_0^2}{gh}\left(\frac{1}{2}\left(\frac{\partial q_1}{\partial x_1}\right)^2 + \frac{\partial q_2}{\partial x_1} - \frac{\sigma^2}{\epsilon} V(1, z_1)\frac{\partial^3 q_1}{\partial x_1^3}\right) = 0 \quad \text{at order } \epsilon^2. \tag{3.11}$$

In (3.3) we have defined the q_n, $n = 1, 2, \ldots$, to be functions of x_1 alone, thus from (3.10) we see that η_1 is similarly a function of x_1 only: it is a constant over each section. With this knowledge we can easily obtain the first-order expression for the cross-sectional area $A(x)$:

$$A(x) = A_0 + \epsilon h B_0 \eta_1(x_1) + O(\epsilon^2).$$

That is,

$$\begin{aligned} A(x)/h^2 &= A_0/h^2 + \epsilon \,.\, B_0/h \,.\, \eta_1(x_1) + O(\epsilon^2) \\ &= a_0 + \epsilon b_0 \eta_1(x_1) + O(\epsilon^2) \\ &= a_0 + \epsilon a_1(x_1) + O(\epsilon^2), \end{aligned} \tag{3.12}$$

say. The integrated continuity equation for Q, equation (2.5), gives

$$a_1 + a_0 q_1' = 0 \quad \text{at order } \epsilon, \tag{3.13}$$

and from this, (3.9) and (3.10) we have

$$\frac{\partial V}{\partial y_1}(1, z_1) = \frac{c_0^2}{gh} = \frac{a_0}{b_0}, \tag{3.14}$$

giving the speed of infinitesimal longitudinal disturbances

$$c_0 = (gha_0/b_0)^{\frac{1}{2}} = (gA_0/B_0)^{\frac{1}{2}},$$

as obtained by Kelland (1839), and giving the surface boundary condition for equation (3.7).

Equations (3.7), (3.8) and (3.14) constitute a well-posed Neumann problem on a constant domain. Solutions to this problem, to give $V(y_1, z_1)$, are unique to within an arbitrary constant; if we can solve the problem for a given cross-section we can obtain the variation of velocity over the cross-section relative to that at some fixed point; to obtain actual velocities we still have to solve for η_1 and q_1.

Introducing the momentum equation (2.7), we eliminate the pressure $p(x, y, z)$ by substituting the energy equation (2.6):

$$S = \iint_A [R - gy + \tfrac{1}{2}(\phi_x^2 - \phi_y^2 - \phi_z^2)]\, dy\, dz, \tag{3.15}$$

which gives

$$S - RA(x) + gM(x) = \frac{1}{2}\iint_A (\phi_x^2 - \phi_y^2 - \phi_z^2)\, dy\, dz, \tag{3.16}$$

where

$$A(x) = \iint_A dy\, dz,$$

the cross-sectional area, and

$$M(x) = \iint_A y\, dy\, dz,$$

the first moment of the cross-section about the z axis. This clearly depends on the choice of co-ordinate origin, however so does the magnitude of R, and it is a simple matter to show that $gM(x) - RA(x)$, and hence (3.16), is independent of the co-ordinate origin.

We already have the integrated continuity equation (2.5) for Q:

$$Q = \iint_A \phi_x \, dy \, dz.$$

The form of (2.5) and (3.16) suggests the subtraction of $\frac{1}{2}Q^2/A(x)$ from (3.16) to eliminate leading terms in the series for ϕ_x so that the equation will be in terms of the dispersive and nonlinear parts of the expansion for ϕ.

Before we make this substitution we have to introduce a symbol for integrated quantities. Thus we introduce an integral operator, denoted by $I(x)$, to represent the integral over the cross-section so that we have, for example,

$$\begin{aligned} I(x)\, V(y_1, z_1) &\equiv \iint_a V(y_1, z_1)\, dy_1\, dz_1 \\ &= \left[\iint_{a_0} + \epsilon \iint_{a_1} + \epsilon^2 \iint_{a_2} + \ldots\right] V(y_1, z_1)\, dy_1\, dz_1 \\ &= I_0 V + \epsilon I_1 V + \epsilon^2 I_2 V + O(\epsilon^3). \end{aligned} \tag{3.17}$$

Substituting the expansion (3.6),

$$\phi = lc_0(x_1 + \epsilon q_1(x_1) + \epsilon^2 q_2(x_1) - \epsilon\sigma^2 V(y_1, z_1)\, q_1''(x_1)) + O(\epsilon^3),$$

into (2.5) and (3.16), after performing the necessary differentiations, integrating symbolically by means of the operator $I(x)$ and then subtracting $\frac{1}{2}Q^2/A(x)$ as suggested, we obtain after lengthy manipulations

$$S - RA(x) + gM(x) - \tfrac{1}{2}Q^2/A(x) = O(\epsilon^3),$$

showing the surprising result that the equation is identically satisfied up to and including terms in ϵ^2. Retracing our steps, adding an extra third-order term for completeness into the expansions, $\epsilon^3 I_3$ into (3.17) and $\epsilon^3\phi_{31} + \epsilon^2\sigma^2\phi_{32} + \epsilon\sigma^4\phi_{33}$ into (3.6), and then substituting gives

$$S - RA(x) + gM(x) - \tfrac{1}{2}Q^2/A(x) = -\tfrac{1}{2}\sigma^2\epsilon^2 c_0^2 h^2 I_V q_1''^2(x_1) + O(\epsilon^4), \tag{3.18}$$

where
$$I_V = \iint_{a_0} (V_{y_1}^2 + V_{z_1}^2)\, dy_1\, dz_1.$$

The variables on the left-hand side are all exact, being in terms of integrals over the entire cross-section of the fluid. The differential term q_1'' is of first order, thus we see that any solutions obtained from this differential equation will be of first-order accuracy only. All second- and third-order terms are satisfied identically to $O(\epsilon^4)$.

Multiplying through by $A(x)$, substituting (3.13), which connects q_1' and a_1, and converting all non-dimensional terms back to their physical counterparts now that the order of accuracy of each term has been recognized, we have

$$\tfrac{1}{2}Q^2 I_{P_1} A_0^{-3} A_*'^2(x) + gM(x)A(x) - RA^2(x) + SA(x) - \tfrac{1}{2}Q^2 = O(\epsilon^4), \tag{3.19}$$

in which $A_*(x)$ is defined by $A(x) = A_0 + A_*(x) + O(\epsilon^2)$ and

$$I_{P_1} = h^4 I_V = \iint_{A_0} (P_{1y}^2 + P_{1z}^2)\, dy\, dz.$$

3.2. *Solution of differential equation*

The coefficient of the differential term may be transformed by the following theorem, easily obtained from the divergence theorem:

$$\iint_{A_0} (\nabla P_1)^2\, dy\, dz = \int_{B_0+C} P_1 \frac{\partial P_1}{\partial n} - \iint_{A_0} P_1 \nabla^2 P_1\, dy\, dz.$$

We have $\nabla^2 P_1 = 1$ throughout, $\partial P_1/\partial n = 0$ on solid boundaries, and

$$\partial P_1/\partial n = A_0/B_0$$

on the surface $y = h$. Therefore

$$\begin{aligned} I_{P_1} &= A_0\left(\frac{1}{B_0}\int_{B_0} P_1(h,z)\, dz - \frac{1}{A_0}\iint_{A_0} P_1(y,z)\, dy\, dz\right) \\ &= A_0(\overline{P}_{1B} - \overline{P}_{1A}). \end{aligned} \tag{3.20}$$

The term I_{P_1} is a positive-definite integral; the arbitrary constant in the solution $P_1(y,z)$ plays no part because the integral is in terms of the gradient of this quantity. Similarly in (3.20) we have the difference between two values, eliminating the constant. Physically, I_{P_1} represents some measure of the physical extent of the cross-section relative to the channel bottom. For a rectangular cross-section, $P_1 = \frac{1}{2}y^2$, giving $I_{P_1} = \frac{1}{3}Bh^3$, the second moment of area of the channel cross-section about the z axis. The function $P_1(y,z)$ has also been obtained for some more general sections: Peters gave the solution for a semicircle, while Peregrine gave that for a triangle: $P_1(y,z) = \frac{1}{4}(y^2+z^2)$. From this solution as well, we see that I_{P_1} is roughly related to some second moment of the cross-section.

From (3.20), we can rewrite the differential equation (3.19) as

$$\tfrac{1}{2}Q^2(\overline{P}_{1B} - \overline{P}_{1A})(A'_*(x)/A_0)^2 + gM(x)\,A(x) - RA^2(x) + SA(x) - \tfrac{1}{2}Q^2 = O(\epsilon^4).$$

This is quite similar to the equation developed by Benjamin & Lighthill for rectangular channels in terms of $\eta(x)$, the free-surface elevation. They obtained

$$\tfrac{1}{6}Q^2\eta'^2(x) + \tfrac{1}{2}g\eta^3(x) - R\eta^2(x) + S\eta(x) - \tfrac{1}{2}Q^2 = 0$$

for a channel of unit width, which is easily obtained from the above equation. The only real complication in the more general case is that the differential term has a coefficient in terms of a Neumann problem on the undisturbed channel cross-section, and we have the first moment of area of the variable cross-section ($M(x)$) in one term. This equation has been obtained by, and still represents, a momentum balance in the channel obtained by integrating the physical quantities over the irregular cross-section. The differential term, as is clearly shown by (3.18), represents the dispersion effects, allowing for velocity variation over the cross-section. Now, we eliminate $M(x)$ by relating it to $A(x)$ to give a differential equation in $A(x)$.

Writing a Taylor expansion for the width B, about the undisturbed level, we have, where $y_* = y-h$,

$$B(y_*) = B_0 + B'_0 y_* + \tfrac{1}{2}B''_0 y_*^2 + O(\epsilon^3).$$

Now the first-order solution η_1 is a constant on any cross-section, from (3.10), hence we may write for the area

$$A = A_0 + \int_0^{\epsilon h\eta_1} B(y_*)\,dy_* + \epsilon^2 A_2 + \epsilon^3 A_3 + O(\epsilon^4),$$

where A_2 and A_3 are contributions to the cross-sectional area by the second- and third-order solutions. Substituting for $B(y_*)$ and integrating, we obtain

$$A = A_0 + \epsilon B_0 h\eta_1 + \epsilon^2(\tfrac{1}{2}h^2\eta_1^2 B_0' + A_2) + \epsilon^3(\tfrac{1}{6}h^3\eta_1^3 B_0'' + A_3) + O(\epsilon^4).$$

Similarly we write, for the first moment about the z axis,

$$\begin{aligned} M &= A_0\bar{y} + \int_0^{\epsilon h\eta_1} (h+y_*)B(y_*)\,dy_* + \epsilon^2 hA_2 + \epsilon^3(hA_3 + hA_2\eta_1) + O(\epsilon^4) \\ &= A_0\bar{y} + \epsilon h^2 B_0\eta_1 + \epsilon^2[\tfrac{1}{2}h^2\eta_1^2(hB_0' + B_0) + hA_2] \\ &\qquad + \epsilon^3[\tfrac{1}{3}h^3\eta_1^3(\tfrac{1}{2}hB_0'' + B_0') + hA_3 + hA_2\eta_1] + O(\epsilon^4). \end{aligned}$$

Now we can invert the series for A, and substituting into this expression gives

$$M = A_0\bar{y} + h(A - A_0) + \frac{1}{2B_0}(A - A_0)^2 - \frac{1}{6}\frac{B_0'}{B_0^3}(A - A_0)^3 + O(\epsilon^4). \qquad (3.21)$$

The terms in A_2, A_3 and B_0'' have disappeared but B_0' remains in the third-order term: we must limit our attention to channels which are not so shallow that this term is of a lower order, and which have B_0'' similarly not large.

Equation (3.21) suggests the use of $A_* = A - A_0$ as the area variable; substituting this and (3.21) into the differential equation (3.19) yields

$$\begin{aligned} &\tfrac{1}{2}Q^2(\bar{P}_{1B} - \bar{P}_{1A})(A_*'/A_0)^2 + A_*^3(g/2B_0 - gB_0'A_0/6B_0^3) \\ &\quad + A_*^2(gh + gA_0/2B_0 - R) + A_*(gA_0h + gA_0\bar{y} - 2RA_0 + S) \\ &\quad + (gA_0^2\bar{y} - RA_0^2 + SA_0 - \tfrac{1}{2}Q^2) = O(\epsilon^4), \end{aligned} \qquad (3.22)$$

giving a differential equation for $A_*(x)$ in terms of the invariants of the flow Q, R and S, and the geometrical parameters of the channel A_0, B_0, B_0', h, $\bar{y}$ and $\bar{P}_{1B} - \bar{P}_{1A}$. We have now found the direct equivalent of the Benjamin & Lighthill equation, but with more complicated coefficients; importantly, the non-differential terms are in the form of a cubic, so that we can use the same physical arguments as Benjamin & Lighthill. This will be done in § 4, meanwhile we study the terms of the equation as follows.

Multiplying through by B_0/gA_0^3, we can write (3.22) as

$$\begin{aligned} &\tfrac{1}{2}(Q^2B_0/gA_0^3)(\bar{P}_{1B} - \bar{P}_{1A})(A_*'/A_0)^2 + (A_*/A_0)^3(\tfrac{1}{2} - \tfrac{1}{6}B_0'A_0/B_0^2) \\ &\quad + (A_*/A_0)^2(\tfrac{1}{2} + hB_0/A_0 - RB_0/gA_0) \\ &\quad + (A_*/A_0)(hB_0/A_0 + \bar{y}B_0/A_0 - 2RB_0/gA_0 + SB_0/gA_0^2) \\ &\quad + (\bar{y}B_0/A_0 - RB_0/gA_0 + SB_0/gA_0^2 - \tfrac{1}{2}Q^2B_0/gA_0^3) = O(\epsilon^4). \end{aligned} \qquad (3.23)$$

We have made the equation dimensionless, in terms of the following quantities.

(*a*) Variables:

$$\mathscr{A} = A_*/A_0, \quad \xi = xB_0/A_0.$$

(*b*) Geometrical parameters of the channel cross-section:

$\theta_1 = B_0' A_0/B_0^2$, a measure of the bank slope at the undisturbed waterline.

$\theta_2 = hB_0/A_0$, a measure of the non-rectangularity of the cross-section by area.

$\theta_3 = (h-\bar{y})/(\frac{1}{2}A_0/B_0)$, a measure of non-rectangularity by the first moment of area about the undisturbed waterline.

$\theta_4 = (\bar{P}_{1B}-\bar{P}_{1A})/(A_0/B_0)^2$, in terms of the solution of the Neumann problem, which is some measure of the second moment of the cross-section relative to the bottom.

(*c*) Physical quantities associated with the flow:

$$q = Q(B_0/gA_0^3)^{\frac{1}{2}}, \quad r = RB_0/gA_0, \quad s = SB_0/gA_0^2.$$

Throughout this non-dimensionalization, as throughout this work, the fundamental importance of the hydraulic mean depth is obvious.

On examining (3.23) we see that it is convenient to redefine r and s as

$$r = \frac{B_0}{gA_0}(R-gh), \quad s = \frac{SB_0}{gA_0^2} - \frac{B_0}{A_0}(h-\bar{y}).$$

Substituting the newly defined quantities into (3.23) we have

$$q^2\theta_4(d\mathscr{A}/d\xi)^2 + \mathscr{A}^3(1-\tfrac{1}{3}\theta_1) + \mathscr{A}^2(1-2r) + \mathscr{A}(2s-4r) + (2s-2r-q^2) = O(\epsilon^4). \quad (3.24)$$

This equation will be discussed more fully in §4; at this point we proceed to an analytical solution.

If the cubic in (3.24) has the roots $\gamma_1 \geqslant \gamma_2 \geqslant \gamma_3$, we obtain one of the following two solutions, depending on the sign of the coefficient of $\mathscr{A}^3$. When this is positive, the more usual case, we have

$$\mathscr{A} = \gamma_2 + (\gamma_1-\gamma_2)\,\mathrm{cn}^2(\alpha\xi, k),$$

where
$$k = \left(\frac{\gamma_1-\gamma_2}{\gamma_1-\gamma_3}\right)^{\frac{1}{2}}, \quad \alpha = \frac{1}{2}\left[\frac{(\gamma_1-\gamma_3)(1-\frac{1}{3}\theta_1)}{q^2\theta_4}\right]^{\frac{1}{2}}. \quad (3.25)$$

The other case, when $\theta_1 > 3$, is given by

$$\mathscr{A} = \gamma_2 - (\gamma_2-\gamma_3)\,\mathrm{cn}^2(\alpha\xi, k),$$

$$k = \left(\frac{\gamma_2-\gamma_3}{\gamma_1-\gamma_3}\right)^{\frac{1}{2}}, \quad \alpha = \frac{1}{2}\left[\frac{(\gamma_1-\gamma_3)(\frac{1}{3}\theta_1-1)}{q^2\theta_4}\right]^{\frac{1}{2}}. \quad (3.26)$$

Having obtained solutions to the governing differential equation, the cross-sectional area has less significance than the elevation of the free surface, so we convert, using the first-order relationship

$$\epsilon h\eta_1(x_1) = \eta_*(x) = (A_0/B_0)\mathscr{A}(\xi),$$

showing that to this order the free-surface elevation varies as $\mathrm{cn}^2(\alpha\xi, k)$, giving a cnoidal wave form.

The second case (3.26) is a wave of depression, having the usual cn^2 shape, but with sharper troughs than crests, showing that the normal tendency of nonlinear waves to have sharper crests has been more than offset by the geometrical characteristics of the channel. Peregrine (1968) has given examples of channels having this property.

To the first-order accuracy of the present analysis there is no change in the free-surface elevation across the channel. We can, however, obtain the second-order cross-channel variation using the equations already derived. In (3.11), which is valid for all z_1 on the surface $y_1 = 1$, only η_2 and $V(1, z_1)$ are functions of z_1. Thus we separate η_2 into constant and variable parts on a particular cross-section:

$$\eta_2 = \eta_{21}(x_1) + \eta_{22}(x_1, z_1),$$

and, equating the terms in (3.11) which are functions of z_1,

$$\eta_{22}(x_1, z_1) = \frac{c_0^2}{gh}\frac{\sigma^2}{\epsilon} V(1, z_1)\, q_1''',$$

giving

$$\epsilon^2 \eta_{22}(x_1, z_1) = -\epsilon \eta_1'' \sigma^2 V(1, z_1).$$

Thus we have

$$\eta = h + \epsilon h \eta_1 + \epsilon^2 h \eta_{22} + O(\epsilon^2),$$

which, converting to physical variables, gives

$$\eta(x, z) = h + \eta_*(x) - P_1(h, z)\, \eta_*''(x) + O(\epsilon^2), \tag{3.27}$$

where $\eta_* = \mathscr{A} A_0/B_0$, obtained from (3.25) or (3.26). Thus we have a first-order solution $\eta_*(x)$ which is constant across the channel, and varies as $\mathrm{cn}^2(\alpha\xi, k)$ along the channel, and a second-order solution which gives the variation across the channel. At the order of approximation used throughout the analysis, $\eta_{21}(x_1)$ cannot be obtained, so the error term is of second order as shown. The solution $P_1(h, z)$ includes an arbitrary constant: we set this so that the minimum value of P_1 on the surface is zero; all cross-channel variation occurs relative to this point.

4. Discussion

4.1. *Physical interpretation of the governing differential equation*

While the solutions (3.25) and (3.26) show the variation of the free-surface elevation, the role of the physical flow invariants is not as explicit as when examined using the differential equation (3.24). We set out to do this, following the discussion of Benjamin & Lighthill (1954), who studied cnoidal waves and bores in rectangular channels.

Considering channels of arbitrary cross-section, we note that $\mathscr{A}$ is a monotonically increasing function of wave height, so that an increase in $\mathscr{A}$ means an increase in η, and we naturally refer to it as a higher wave.

If we substitute $\mathscr{P}_3(\mathscr{A})$ for the polynomial in (3.24), we can write

$$q^2\theta_4 (d\mathscr{A}/d\xi)^2 + \mathscr{P}_3(\mathscr{A}) = 0, \tag{4.1}$$

where

$$\mathscr{P}_3(\mathscr{A}) = \mathscr{A}^3(1 - \tfrac{1}{3}\theta_1) + \mathscr{A}^2(1 - 2r) + \mathscr{A}(2s - 4r) + 2s - 2r - q^2, \tag{4.2}$$

noting that the error term $O(\epsilon^4)$ is ignored henceforth. We can also write $\mathscr{P}_2(\mathscr{A})$ for the quadratic terms:

$$\mathscr{P}_3(\mathscr{A}) = \mathscr{P}_2(\mathscr{A}) + \mathscr{A}^3(1 - \tfrac{1}{3}\theta_1).$$

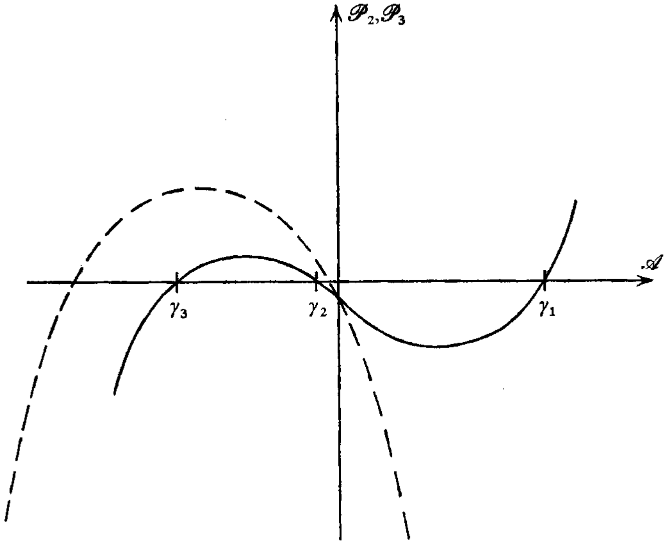

FIGURE 1. Typical $\mathscr{P}_2$ (broken line) and $\mathscr{P}_3$ (solid line) curves for given channel geometry and flow invariants.

From (4.1) we can only have solutions when $\mathscr{P}_3 \leqslant 0$, and we can only have waves when there are two distinct roots, between which $\mathscr{P}_3(\mathscr{A})$ is negative. This is shown by figure 1, on which $\mathscr{P}_2(\mathscr{A})$ and $\mathscr{P}_3(\mathscr{A})$ are plotted for a particular channel and particular q, r and s. The only region for which waves are possible is that contained between the roots γ_1 and γ_2, corresponding respectively to crests and to troughs. For $\mathscr{A} \leqslant \gamma_3$, we satisfy the condition $\mathscr{P}_3 \leqslant 0$, but we cannot have solutions, for this corresponds to the depth becoming zero in a finite distance.

Using figure 1, the role of the coefficient $1-\frac{1}{3}\theta_1$ of $\mathscr{A}^3$ becomes more clear. If this is positive, we have $\mathscr{P}_3$ as shown; if it is zero, then $\mathscr{P}_3$ becomes $\mathscr{P}_2$ and no periodic solutions are possible, for the gravitational nonlinearities are cancelled by the channel properties, and there are no two roots between which $\mathscr{P}_3$ is negative. Then, if $1-\frac{1}{3}\theta_1 < 0$, we shall have a cubic which has $\mathscr{P}_3$ negative between roots γ_2 and γ_3, and we shall have waves where the troughs are sharper than the crests, and if a solitary wave exists it will be one of depression, not elevation. Conditions where this occurs, where $B_0' A_0/B_0^2 > 3$, are for channels with a large area but small surface width and finite surface slope, as given by Peregrine (1968). As this type of channel is seldom encountered, we shall not consider it again; in any case the arguments produced here can be used for this type as well.

The coefficient $1-2r$ of $\mathscr{A}^2$ has some significance: if it is zero, $\mathscr{P}_2$ reduces to a straight line and $\mathscr{P}_3$ has only one root, corresponding to uniform flow. As $r = \frac{1}{2}U^2 B_0/gA_0$ for all uniform flows, in this case we obtain $U^2 = gA_0/B_0$, defining critical flow, on which no waves are possible and for which $\mathscr{A} \equiv 0$.

Similarly we can study the coefficients of $\mathscr{A}^1$ and $\mathscr{A}^0$ for the uniform flow case. From the invariant equations (2.5)–(2.7), written for uniform flow and non-dimensionalized as above,

$$q = F, \quad r = \tfrac{1}{2}F^2, \quad s = F^2, \tag{4.3}$$

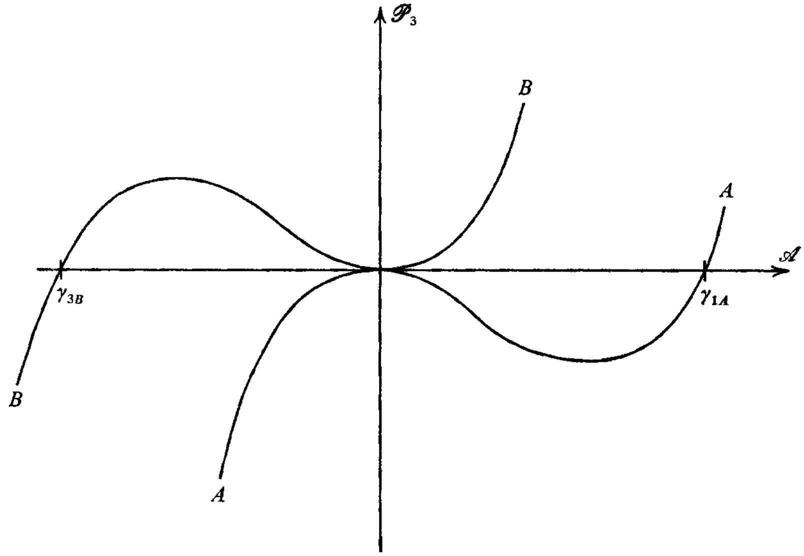

FIGURE 2. $\mathscr{P}_3$ curves for uniform flow: AA, supercritical flow and the solitary wave; BB, subcritical flow.

where F is the Froude number defined for channels of arbitrary cross-section:

$$F = U(B_0/gA_0)^{\frac{1}{2}}.$$

If we substitute (4.3) into (4.2) the coefficients of both $\mathscr{A}^1$ and $\mathscr{A}^0$ disappear:

$$\mathscr{P}_3(\mathscr{A}) = \mathscr{A}^2[1-F^2+\mathscr{A}(1-\tfrac{1}{3}\theta_1)], \tag{4.4}$$

and we have the roots

$$\mathscr{A} = 0 \text{ (repeated)}, \quad \mathscr{A} = (F^2-1)/(1-\tfrac{1}{3}\theta_1). \tag{4.5}$$

As we do not treat $\theta_1 > 3$ here, we have two cases as shown in figure 2, depending on whether $F \gtrless 1$.

If $F > 1$, we have the curve AA, with $\mathscr{A} = 0$ corresponding to uniform supercritical flow. However, the other root γ_{1A} is possible, and we have a solitary wave rising out of the uniform flow. If the amplitude of this wave is a_m, then we have the first-order geometrical relationship $\mathscr{A} = a_m B_0/A_0$, and obtain from (4.5) the first-order expression for wave speed

$$U = (gA_0/B_0)^{\frac{1}{2}}\left(1+\tfrac{1}{2}a_m\frac{B_0}{A_0}(1-\tfrac{1}{3}\theta_1)\right)+O(\epsilon^2) \tag{4.6}$$

and solving (4.1) and (4.4), then converting to η, the height of the free surface,

$$\eta = h+a_m\,\mathrm{sech}^2\frac{x}{2}\left[a_m\frac{B_0}{A_0}\frac{1-\frac{1}{3}\theta_1}{\overline{P}_{1B}-\overline{P}_{1A}}\right]^{\frac{1}{2}}+O(\epsilon^2). \tag{4.7}$$

This result agrees with Peters (1966) and Peregrine (1968), and is the particular case of (3.25) for uniform supercritical flow, when $\gamma_2 = \gamma_3 = 0$, giving $k = 1$, with $\mathrm{cn}^2\,(\alpha\xi, 1) = \mathrm{sech}^2\,\alpha\xi$.

The other case, when $F < 1$, is for uniform subcritical flow, when we have the curve BB shown in figure 2. Root γ_{3B} is not possible, leaving only the root $\mathscr{A} = 0$, corresponding to uniform flow: finite amplitude waves are not possible.

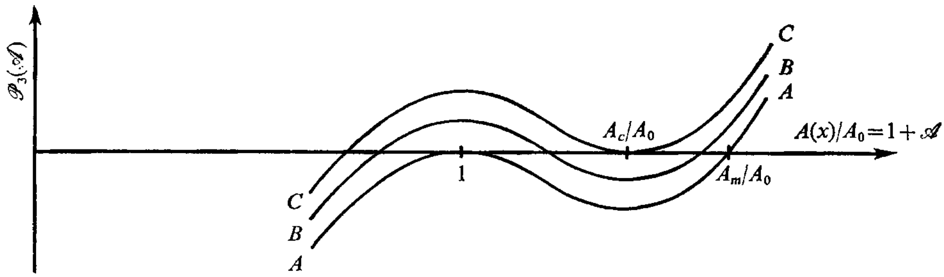

FIGURE 3. Typical $\mathscr{P}_3$ curve for a given channel, showing vertical displacements caused by energy and momentum changes. *AA*, uniform supercritical flow and the solitary wave; *BB*, periodic waves of cnoidal form; *CC*, uniform subcritical flow.

4.2. *Energy and momentum changes for waves and uniform flows*

The governing differential equation (3.24) has been obtained in terms of energy, momentum and discharge. Considering a flow system as a whole, it is of considerable use in predicting the type of flow or wave system generated on a flow by bringing about energy and momentum changes. Examples are bores and hydraulic jumps, barriers across the flow such as sluice gates, and changes of cross-section. These are all examples where the change occurs rapidly, in which case we can quantitatively predict downstream conditions for given upstream conditions in terms of intermediate changes, and vice versa. Even in situations where the changes occur gradually, we can use the type of plot shown in figures 1 and 2 to describe the flow qualitatively.

To illustrate use of the equation, we consider the case of a bore, where we have an energy loss in a flow of constant momentum and discharge. Figure 3, in which the vertical axis is displaced by unity so that the abscissa is $A/A_0 = 1+\mathscr{A}$, is otherwise the same plot as figures 1 and 2. The limitation for the theory to be applicable is that waves should be small, that is, $A_m - A_0 \ll A_0$.

Considering a co-ordinate system in which the bore is stationary, we have uniform supercritical flow upstream, as defined by curve *AA* touching the axis at $A = A_0$. As we have seen, the only wave possible is a solitary wave, and we do not have a proper bore.

If we lose some energy, however little, at the bore, then the effect will be to raise the curve to one like *BB* in the figure. This is because the coefficient of r is $-2(1+\mathscr{A})^2$, which is always negative, hence a reduction in r raises the curve. Immediately we have the general case, with the free surface defined by (3.25), giving periodic waves of cnoidal form, with the maximum and minimum depth defined by the roots of *BB* in figure 3. Importantly, as we shall see later, the maximum and minimum are bounded by the limits of the solitary wave. If we lose more energy, the curve is raised further, and the wave amplitude is reduced further: if the maximum amount possible is lost, we have the uniform subcritical stream of area A_c, given by the repeated root of curve *CC*, which is between A_0 and A_m.

Thus, a cnoidal wave train can be present behind the bore provided that the energy dissipated at the bore is between zero and the value corresponding to a complete hydraulic jump.

Experimental observations (Sandover & Taylor 1962; Peregrine 1969) have shown that waves in channels with appreciably sloping sides have a tendency to break at the edges, thus if we have a wave system with amplitude defined by the roots of BB in figure 3, there may be a continuing tendency for the waves to break at part or all of the crest, losing energy, thereby raising the curve and reducing the wave amplitude until no further breaking occurs.

The above analysis, which predicts the wave form downstream of a bore in terms of the energy loss at the bore, which may be anywhere between zero and that of a complete hydraulic jump, can be used to explain the scatter in results obtained experimentally by Benet & Cunge (1971). They plotted experimental results for different values of the ratio $U_0/(gh)^{\frac{1}{2}}$, where U_0 is the *absolute* velocity of flow upstream, and found less scatter where this was zero or small. Also, they noted that the critical breaking condition closely depended on the initial flow, as measured by the above ratio. This is a clear indication that energy loss in a bore is closely related to the upstream flow conditions. We can imagine that a bore progressing onto still water, where the absolute velocity of the fluid anywhere in the system is small, is much more likely to propagate with little breaking or loss than the case of a bore on running water, in which surface irregularities and roughness and real-fluid effects all contribute to the development of energy losses, breaking and the variability of results. The extreme case is that of a stationary bore on supercritical flow emerging from, say, a sluice gate, with a greatly enhanced tendency to breaking and loss.

Accepting then, the variability of energy losses in bores, we explain the scatter of recorded wave amplitudes; we have closely related energy loss to wave form for given momentum and discharge. The present theory makes no prediction of these losses for given upstream conditions, but it does enable the downstream waves to be calculated for a given energy loss. This may be compared with the theory of Preissmann & Cunge (1967), which assumes that in all bores the energy loss is the maximum possible, that based on hydraulic jump theory, and that any wave system downstream, assumed to be sinusoidal, must possess all of this energy. No allowance is made for variability in the amount of energy loss, and hence no explanation can be given for the scatter in experimental results.

In this discussion of the applications of the present theory, we have limited our attention to a uniform supercritical flow subject to a finite energy loss. The theory, and figure 3, are more general than this, and we can study the behaviour of uniform flows or of systems of waves in which energy or momentum changes occur. If a momentum decrease occurs, for example, when uniform subcritical flow passes over a positive step or some other obstacle in the flow, then because the coefficient of s in $\mathscr{P}_3$ is $2(1+\mathscr{A})$, always positive, the curve is lowered and we have the development of a wave system. When we cause enough loss, as for a sluice gate across the flow, then we pass from uniform subcritical flow to uniform supercritical flow. If the downstream conditions are such that this flow cannot be supported, if the depth is too great, say, there will be some kind of bore, in which case the technique is re-applied to estimate the nature of the flow downstream, and so on.

4.3. *Practical design note*

We include a practical note, concerning the design of channels of non-rectangular cross-section in which stationary or travelling bores are expected to occur: the present theory shows a simple design criterion.

As the critical design parameter is the amount of freeboard at the sides of the channel, governing the amount of excavation or materials used, we are primarily concerned with the maximum water level, rather than wavelength or wave speed. From figure 3 we see that the highest water level possible is that at the crest of a solitary wave, corresponding to a bore with no energy loss. For this wave we have a first-order expression for the amplitude:

$$a_m = \frac{A_0}{B_0}\frac{U^2B_0/gA_0-1}{1-\frac{1}{3}B_0'A_0/B_0^2}. \tag{4.8}$$

Thus we have a simple estimate of the necessary freeboard in terms of the Froude number, hydraulic mean depth and a dimensionless measure of the side slope. This concept, of using the solitary wave as the design criterion, is borne out by the results of Sandover & Taylor, who noted that channel friction had little effect on the height of the first wave in an undular bore, indicating that losses before the first wave had been small, showing that it is close to a solitary wave.

A factor to be considered in the design of non-rectangular channels is the variation of the free-surface elevation across the cross-section as given by (3.27). For the type of regular cross-section encountered in practice (e.g. trapezoidal), $P_1(h, z)$ appears to be a well-behaved function which has a minimum at the centre of the channel and increases towards the banks (Peregrine 1968, 1969). From (3.27) we see that for wave crests, where η_* is a maximum and η_*'' has a maximum negative value, that the highest surface elevation occurs at the sides of the channel, providing the design criterion. The quantity η_*'' is greater for periodic waves than for the solitary wave, but as it is a second-order contribution we can continue to consider the solitary wave as the worst case.

Differentiating (4.7), the equation for the solitary wave, and substituting into (3.27) we have

$$\eta_{\text{crest}}(z) = h + a_m + \tfrac{1}{2}a_m^2\frac{B_0}{A_0}\frac{1-\frac{1}{3}B_0'A_0/B_0^2}{\overline{P}_{1B}-\overline{P}_{1A}}P_1(h,z), \tag{4.9}$$

giving the variation across the crest in terms of the solution $P_1(y, z)$. An important limitation to this consideration of cross-channel variation is that the channel be not broad, which will be discussed in the next section. For most types of channel encountered in practice, however, (4.8) and (4.9) give a rational design criterion for the necessary depth of channel based on the highest possible waves for a given operating discharge and channel geometry.

4.4. *Limitations of the theory*

The nonlinear dispersive wave theory presented in § 3 has several limitations, both mathematical and practical. We can only use the theory to describe wave systems which are of small amplitude and which are shallow (where the depth of flow is small compared with the wavelength).

Applications of this theory are limited to channels which are of a regular cross-section and which are not broad. Peregrine (1968) has shown that for broad channels with sloping sides, the cross-channel variation is of such a magnitude that one of the major assumptions (that wave amplitude is small) is violated.

Conditions at the bank are complicated by two more factors. The waves may break at the banks, giving local energy dissipation, which cannot be studied by the present theory. Also, because the local depth is small, nonlinearity is more important near the bank. This is a cause of some of the effects noted experimentally (Sandover & Taylor 1962; Peregrine 1969), for which the present work can offer no explanation. These effects include the tendency for waves to be unsteady in broad channels, and for the crests to be curved in plan. The occurrence of a 'fish-tail' wave pattern, which almost obscures the main waves, remains similarly unexplained.

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